Yiddish of Day

"Oyb me vill di Kneydlekh, zog di haggadah"

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אויב מע וויל די
קניידלעך,
זאָג די הגדה!

"if you want to eat matzo balls, you must read the Haggadah"

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Symmetric / Alternating Products

Def: Suppose V, W are $\mathbb{F}$-vs. Then a multilinear function $f: \underbrace{V \times \cdots \times V}_{k\text{-times}} \to W$ is

1) <u>Symmetric</u> if $f(v_1, v_2, \ldots, v_k) = f(v_k, v_1, v_2, \ldots, v_{k-1})$ $= \cdots$

= the same if I permute the order

ex) $f: V \times V \to W$ $\quad f(v_1, v_2) = f(v_2, v_1)$

2) <u>alternating</u> if $f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$

ex) <u>$k=2$</u>: $f: V \times V \to W$ is alternating iff $f(v,v) = 0$

ex) Suppose $f: V \times V \to W$ is alternating. Then

$f(v_1, v_2) = -f(v_2, v_1)$ (this is called skew-symmetric)

Pf) (maybe HW?) Yes

(Rmk: True more generally than 2 vector spaces)

We saw before that a bilinear map

$$f : V \times V \longrightarrow W$$

is the "same thing" as a linear map

$$\tilde{f} : V \otimes V \longrightarrow W$$

Now, what if this f was alternating?

- if $v_1 = v_2$ we have

$$0 = f(v_1, v_2) = \tilde{f}(v_1 \otimes v_2)$$

- That is the subspace

$$N := \langle v \otimes v \mid v \in V \rangle \subseteq \ker(\tilde{f})$$

will be contained in the <u>kernel</u> of $\tilde{f}$

\bigwedge

- We consider the <u>quotient</u> and denote it by $\Lambda^2(V) := V \otimes V / N$

(call this the (second) exterior power of V)

- the coset of $v_1 \otimes v_2$ in $\Lambda^2(V)$ is denoted

<u>$v_1 \wedge v_2$</u> ("v_1 wedge v_2")

Why? Universal Prop of quotient! We saw above that if

$$f : V \times V \longrightarrow W$$

is alternating

then the induced map

$$\tilde{f} : V \otimes V \longrightarrow W$$

contains N in $\ker(\tilde{f})$

$\rightsquigarrow$ We get a well defined map

$$\tilde{f} : \Lambda^2(V) \longrightarrow W$$

__Thm:__ The composite

$$V \times V \longrightarrow \underline{V \otimes V} \longrightarrow \underline{\Lambda^2(V)}$$

is __alternating__.

Moreover, given __any__ alternating map

$$f: V \times V \longrightarrow W$$

there's a __unique__ __linear__ map

$$\tilde{f}: \underline{\Lambda^2(V)} \longrightarrow W$$

such that

$$\begin{tikzcd} V \times V \arrow[r, "f"] \arrow[d] & W \\ \Lambda^2(V) \arrow[ur, dashed, "\exists!\, \tilde{f}"'] & \end{tikzcd}$$

(ie $\tilde{f}(v_1 \wedge v_2) = f(v_1, v_2)$)

(that is there's a bijection

$$\mathrm{Alt}(V \times V, W) \cong \mathcal{L}(\Lambda^2(V), W)$$
)

So to define a map out of $\underline{\Lambda^2(V)}$

one just defines a bilinear map $V \times V \to W$ and verifies

it is alternating

ex) Prove there is a unique linear map

$$\Lambda^2(V) \longrightarrow \underline{V \otimes V}$$

that sends $v_1 \wedge v_2 \longmapsto v_1 \otimes v_2 - v_2 \otimes v_1$

Pf) Define $f : V \times V \to V \otimes V$

$$f(v_1, v_2) = v_1 \otimes v_2 - v_2 \otimes v_1$$

Verify f is bilinear and alternating.

$$f(v_1 + v_1', v_2) = (v_1 + v_1') \otimes v_2 - v_2 \otimes (v_1 + v_1')$$
$$= v_1 \otimes v_2 + v_1' \otimes v_2 - v_2 \otimes v_1 - v_2 \otimes v_1'$$
$$= f(v_1, v_2) + f(v_1', v_2)$$

Also $f(v, v) = v \otimes v - v \otimes v = 0$

$\leadsto \exists!\ \tilde{f} : \Lambda^2(V) \to V \otimes V$ st $\tilde{f}(v_1 \wedge v_2) = v_1 \otimes v_2 - v_2 \otimes v_1$

Questions

1) What does it mean that $v_1 \wedge v_2 = 0$?

A: $v_1 \wedge v_2 = 0 \iff$ every alternating bilinear map $f: V \times V \to W$ has $f(v_1, v_2) = 0$

2) What does it mean that $\Lambda^2(V) = 0$?

A: $\Lambda^2(V) = 0 \iff$ every bilinear alternating map $f: V \times V \to W$ vanishes everywhere ($f(v_1, v_2) = 0 \quad \forall v_1, v_2$)

1) What does it mean to say that $a \otimes b = 0$

A: $a \otimes b = 0 \iff$ every bilinear map $f: V \times V \to W$ must have $f(a,b) = 0$

Pf) "$\Leftarrow$" Assume every bilinear map $f: V \times V \to W$ has $f(a,b) = 0$. Then $a \otimes b = \pi(a,b) = 0$

"$\Rightarrow$" Assume $a \otimes b = 0$ and let $f: V \times V \to W$ be a bilinear map

$$
\begin{aligned}
f(a,b) &= \tilde{f}(a \otimes b) \\
&= \tilde{f}(0) \\
&= 0
\end{aligned}
$$

3) What does it mean to say

$$v_1 \wedge v_2 = v_1' \wedge v_2' \ ?$$

A: $v_1 \wedge v_2 = v_1' \wedge v_2' \iff$ every alternating bilinear map $f: V \times V \to W$ has $f(v_1, v_2) = f(v_1', v_2')$

Rmk: All this works for multilinear alternating maps

$$f: \overbrace{V \times \cdots \times V}^{k\text{-times}} \longrightarrow W$$

$\rightsquigarrow$ get the space $\Lambda^k(V)$ for all k

ex) Suppose V is n-dim. Show in HW that if $k > n$

HW then any alternating map

$$f : \underbrace{V \times \cdots \times V}_{k\text{-times}} \longrightarrow W \text{ is } 0$$

$\Longrightarrow$ if $k > \dim V$ then $\Lambda^k(V) = 0$

Q: What if $k = \dim V$?

ex) Suppose V is 2-dim with basis $\mathcal{B}_V = (v_1, v_2)$.

"Lets compute the "elementary wedge""

$$(av_1 + bv_2) \wedge (cv_1 + dv_2)$$

$$= \cancel{ac(v_1 \wedge v_1)}^{0} + ad\, v_1 \wedge v_2 + bc\, v_2 \wedge v_1 + \cancel{bd\, v_2 \wedge v_2}^{0}$$

$$v_1 \wedge v_2 = -v_2 \wedge v_1$$

$$\rightsquigarrow = ad(v_1 \wedge v_2) - bc(v_1 \wedge v_2)$$

$$= \underline{(ad - bc)}\, v_1 \wedge v_2$$

interesting.....

Wedge Products (cont)

- We saw above that, if $k > \dim V$ then

$$\Lambda^k(V) = 0$$

(compare to $\dim(\overbrace{V \otimes \cdots \otimes V}^{k\text{-times}}) = \underline{\dim(V)^k}$)

- We saw that, for a 2-dim vs with basis (v_1, v_2)

wedge $\quad (av_1 + bv_2) \wedge (cv_1 + dv_2)$

$= (ad - bc)\, v_1 \wedge v_2$

Thm: (Notes) Let $\dim V = n$. Then

$$\dim \Lambda^k(V) = \underline{\binom{n}{k}} \qquad \text{"n choose k"}$$

If $\mathcal{B} = (v_1 \ldots v_n)$ is a basis for V then

$$\mathcal{C} = (v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \le i_1 \le \cdots \le i_k \le n)$$

is a basis for $\Lambda^k(V)$

Cor: 1) If $\mathcal{B} = (v_1 \ldots v_n)$ basis for V then

$$\dim(\Lambda^2(V)) = \underline{\binom{n}{2}}$$

with basis

$$\mathcal{C} = \begin{pmatrix} v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, \ldots, v_1 \wedge v_n, v_2 \wedge v_3, \ldots, v_2 \wedge v_n, \\ v_3 \wedge v_4, \ldots, v_3 \wedge v_n, \ldots, v_{n-1} \wedge v_n \end{pmatrix}$$

ex) $\mathcal{B} = (v_1, v_2, v_3)$ then for $\Lambda^2(V)$

$$\mathcal{C} = (v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3)$$

Note that $\dim(\Lambda^2(V)) = 3$

So $\Lambda^2(V) \cong V$

Here's a particular isomorphism in the case $V = \mathbb{R}^3$

ex) $\mathcal{B} = (e_1, e_2, e_3)$ be standard basis for $\mathbb{R}^3$

Define the isomorphism $\Lambda^2(\mathbb{R}^3) \xrightarrow{\sim} \mathbb{R}^3$ by

(☆)

- $e_1 \wedge e_2 \longrightarrow e_3$
- $e_1 \wedge e_3 \longrightarrow -e_2$
- $e_2 \wedge e_3 \longrightarrow e_1$

("Hodge Star operator")

(why is this an iso?)

Then compute $(ae_1 + be_2 + ce_3) \wedge (a'e_1 + b'e_2 + c'e_3)$

$= ab'e_1 \wedge e_2 + ac'e_1 \wedge e_3 - a'be_1 \wedge e_2 + bc'e_2 \wedge e_3$
$- a'ce_1 \wedge e_3 - b'c\, e_2 \wedge e_3$

$= (ab' - a'b)\, e_1 \wedge e_2 + (ac' - a'c)\, e_1 \wedge e_3 + (bc' - b'c)\, e_2 \wedge e_3$

$$\Big\downarrow (\star)$$

$$\begin{pmatrix} bc' - b'c \\ a'c - ac' \\ ab' - a'b \end{pmatrix}$$

Look up formula for <u>cross-product</u> !!

2) $\dim(\Lambda^n(V)) = \underline{1}$ $\qquad \mathcal{B} = (v_1 \ldots v_n)$

with basis

$\mathcal{C} = (v_1 \wedge v_2 \wedge v_3 \wedge \ldots \wedge v_n)$

ex) $\mathcal{B} = (v_1, v_2)$

$\mathcal{C} = (v_1 \wedge v_2)$